A physical system in equilibrium at temperature $T$ has a set of states $\{s\}$ with energy spectrum $E_s$. (You can think of $\{s\}$ as labeling a set of discrete states.)

a) Write down expressions in the canonical ensemble for the

i) normalized probability of finding the system in a state $s$ with energy $E_s$

ii) the canonical partition function $Z$

iii) the entropy $S$, and

iv) the average energy $\langle E \rangle$.

b) Show that

$$-k_B T \ln Z = \langle E \rangle - S T.$$

c) Show that $C_V = \left( \frac{\partial \langle E \rangle}{\partial T} \right)_V$ is proportional to the mean-squared-fluctuation of the energy,

$$\langle E^2 \rangle - \langle E \rangle^2$$

and find the constant of proportionality.
A 2-state quantum mechanical system in thermal equilibrium at temperature $T$ has Hamiltonian

$$H = \begin{pmatrix} \epsilon & \gamma e^{i\delta} \\ \gamma e^{-i\delta} & 0 \end{pmatrix}$$

where $\epsilon$, $\gamma$, and $\delta$ are real constants.

a) Find the energy eigenvalues of the system by finding the roots of the characteristic equation for $H$.

b) Using these eigenvalues, show that the partition function $Z$ can be written as

$$Z = 2e^{-\frac{E}{T}} \cosh \left( \frac{\beta W}{2} \right),$$

and find expressions for $\beta$ and $W$.

c) Find the average energy $<E>$.

d) Find the specific heat at constant volume $C_V$. 
Consider an ideal gas of quantum particles obeying Bose-Einstein statistics. The system has volume $V_n = L^n$ in $n$ dimensions and its particles have an ultra-relativistic linear ($s = 1$) or non-relativistic quadratic ($s = 2$) energy-momentum relation $\varepsilon = C|\vec{p}|^s$.

(a) Determine the $n$- and $s$-dependence of the particle density of states $g(\varepsilon)d\varepsilon$.

(b) Provide the expressions for the average number of particles $N$ and the internal energy $U$ in terms of $\beta = 1/kT$ and fugacity $z = e^{\mu/kT}$.

(c) Provide the expression for the pressure $P$ in terms of $\beta$ and $z$.

(d) Find the relation between $U$ and $PV_n$ in terms of $n$ and $s$. This relation should be proved from the expressions provided in (a – c).

(e) Provide the criterion for yielding Bose-Einstein condensation (BEC). In which of the four cases with $n = 2, 3$ and $s = 1, 2$ can BEC occur?
Consider a dilute gas that can be treated as ideal. The molecules of this gas can exist in two forms, $A$ and $B$, that can interconvert, $A \leftrightarrow B$, such that the two forms exist as an equilibrium mixture. According to the Boltzmann distribution law, the equilibrium ratio of $A$ and $B$ populations is given by

$$\frac{\langle N_A \rangle}{\langle N_B \rangle} = \frac{g_A e^{-\beta \Delta \varepsilon}}{g_B}$$

where $\Delta \varepsilon$ is the energy difference between state $A$ and state $B$, and $g_A$ and $g_B$ are their respective degeneracies.

(a) Derive this result from the condition for equality of the chemical potentials at equilibrium.

The canonical partition function is $Q = q^N / N!$, where $N$ is the total number of particles and $q$ is the Boltzmann weighted sum over all single molecule states, both those associated with $A$ and those associated with $B$.

(b) Show that $Q = \sum_p \exp[-\beta A(N_A, N_B)]$, with $-\beta A(N_A, N_B) = \ln[q_A^{N_A} q_B^{N_B} / \langle N_A \rangle \langle N_B \rangle]$, and where the summation is over all partitions of the $N$ molecules into $N_A$ of type $A$ and $N_B$ of type $B$.

(c) Show that the condition for chemical equilibrium in terms of the chemical potentials is equivalent to minimizing the Helmholtz free energy $\partial A / \partial \langle N_A \rangle = \partial A / \partial \langle N_B \rangle = 0$ subject to the constraint that $\langle N_A \rangle + \langle N_B \rangle = N$ is fixed.

(d) Show that $\langle N_A \rangle = q_A (\partial \ln Q / \partial q_A)_{q_B, N} = N q_A / (q_A + q_B)$ and hence, that

$$\langle N_A \rangle / \langle N_B \rangle = q_A / q_B .$$

(e) Show that the mean square fluctuation in $N_A$ is given by

$$\langle [N_A - \langle N_A \rangle]^2 \rangle = q_A (\partial \langle N_A \rangle / \partial q_A)_{q_B, N} = \langle N_A \rangle / N .$$
Consider an ideal gas of $N$ non-relativistic fermions ($S=\frac{1}{2}$) in a three-dimensional volume $V$ at an absolute temperature $T$.

A. Derive or provide the expression for $n(\varepsilon)d\varepsilon$, representing the number of fermions with energy in the range $\varepsilon$ to $\varepsilon+d\varepsilon$, in terms of the chemical potential $\mu=\mu(T,N/V)$.

B. Determine $n(\varepsilon)$ for $T\rightarrow 0$.

C. Determine $\varepsilon_F=\mu(T=0,N/V)$.

D. Sketch $n(\varepsilon)$ for $kT<\varepsilon_F$.

E. Determine $n(\varepsilon)$ for $\exp(-\mu/kT)=\frac{1}{Z} \gg 1$ in terms of $N$ rather than $Z$.

F. Sketch $n(\varepsilon)$ for $\exp(-\mu/kT)=\frac{1}{Z} \gg 1$.

G. Find the most probable energy $\varepsilon^*$ for $T\rightarrow 0$. Find the corresponding maximum $n(\varepsilon^*)$ in terms of $N$ and $\varepsilon_F$.

H. Find the most probable energy $\varepsilon^*$ for $\exp(-\mu/kT)=\frac{1}{Z} \gg 1$. Find the corresponding maximum $n(\varepsilon^*)$ in terms of $N$ and $kT$.

I. Plot and compare on the same graph $n(\varepsilon)$ vs. $\varepsilon$ the two cases: $n(\varepsilon)$ for $T=0$ and $n(\varepsilon)$ for $T>\frac{\varepsilon_F}{k}$. Which is larger and why?
The Hamiltonian of a two-dimensional dipolar molecule of mass $m$, moment of inertia $I$, and dipole moment $\mu$, in an electric field $E$ is

$$H = \frac{1}{2m} (p_x^2 + p_y^2) + \frac{\mu E}{2I} \cos \theta$$

where $\theta$ is the angle between $\vec{E}$ and $\vec{\mu}$.

a) Find the canonical partition function of a system of $N$ such classical dipolar molecules confined to an area $A$.

b) Obtain low-temperature and high-temperature approximations for the specific heat at constant area, $C_A$, and confirm that the effect of the dipole-field interaction becomes negligible when $\mu E << kT$.

c) The probability of finding a two-dimensional dipolar molecule in a state described by the phase space variables $(x, y, p_x, p_y, \theta)$ is proportional to $e^{\beta H}$. Show that if the electric field is not uniform, then the gas is more dense where the field is higher.

Hints: The modified Bessel function $I_0(u)$ is a monotonically increasing function of $|u|$, with an integral representation:

$$I_0(u) = \frac{1}{\pi} \int_0^\pi e^{uv} \cos \theta d\theta$$

Its power series expansion is:

$$I_0(u) = \sum_{k=0}^{\infty} \frac{(u/2)^{2k}}{(k!)^2}$$

while for large $u$:

$$I_0(u) \sim e^u / \sqrt{2\pi u}.$$
Consider a surface with \( N_0 \) absorption sites, in equilibrium with an ideal gas. Each site can absorb at most one molecule from the gas, binding it with an energy \( -\varepsilon_0 \) < 0.

a) Assuming that \( N \) molecules are absorbed on the surface, determine the canonical partition function \( Q_N(\beta) \) for the corresponding surface system.

**Hint:** First show that the single \( E = -N\varepsilon_0 \) energy level has a degeneracy \( g(N) = \frac{N_0!}{N!(N_0 - N)!} \).

b) Determine the grand-canonical partition function \( Z(z, \beta) = \sum_{N=0}^{N_0} z^N Q_N(\beta) \) for the surface system, in terms of the surface fugacity \( z = \exp(\beta \mu) \), where \( \mu \) is the chemical potential and \( \beta = 1/kT \).

c) Determine the average number \( \bar{N} = z \frac{\partial}{\partial z} (\log Z)_{\beta} \) of molecules absorbed on the surface, and the corresponding covering ratio \( \theta = \frac{\bar{N}}{N_0} \), in terms of \( z \).

d) Determine the grand-canonical partition function \( Z_g(z_g, \beta) \) for the ideal gas, in terms of the gas fugacity \( z_g \).

e) Determine the average number \( \bar{N}_g \) of molecules in the gas, in terms of \( z_g \).

f) What is the condition of equilibrium between the gas and the surface system? From that, derive the Langmuir isotherm \( \theta = \frac{P}{P + f(T)} \), where \( P \) is the gas pressure and \( f(T) \) is a function of the absolute temperature, which you must determine.
For a Classical Ideal Gas at a temperature $T$, provide the (Maxwell)

a) Probability $P(v)dv$ that any one molecule has a speed between $v$ and $v + dv$. Include the proper normalization.

b) Determine the most probable speed $v^\ast$, average speed $\langle v \rangle$, and root-mean-square speed $v_{rms} = \sqrt{\langle v^2 \rangle}$. Also determine the standard deviation $\sigma = \sqrt{\langle v^2 \rangle - \langle v \rangle^2}$.

c) For a given $T = T_1$, plot $P(v)$ as a function of $v$. Report $v^\ast$, $\langle v \rangle$, and $v_{rms}$ on the graph. How does that plot scale with increasing $T$? Plot $P(v)$ again for another $T = T_2 > T_1$.

d) At which high temperature $T_c$ might there be roughly a 2.5% fraction of molecules that would have speeds greater than the speed of light? Consider that for a molecule in $H_2$ gas, $mc^2 \approx 1.876 \text{ MeV}$, while $k_B T_0 \approx \frac{1}{40} eV$ at $T_0 = 290 \text{ K}$.
The Debye theory treats a crystal as a continuum, and hence the phonon dispersion relation is $\omega = ck$, where $c$ is the speed of sound in the body. In a ferromagnetic solid at low temperatures, there also exist quantized waves of magnetization (magnons) for which the dispersion relation is of the form $\omega = bk^2$, where $b$ is a constant. In both these cases a large-$k$ (small wavelength) cutoff must be imposed. Take the cutoffs in these two cases as $k_D$ and $k_m$, respectively.

Obtain expressions valid near $T=0$ for the phonon and magnon contributions to the specific heat.

*Note: Your answers may include a dimensionless multiplicative constant in the form of an integral that you *need not* to evaluate. The main point is to obtain the temperature dependence of the two contributions to the specific heat at low temperatures.*
Consider an Ising spin system consisting of three spins \( s = \frac{1}{2} \) on a line with nearest-neighbor interactions. Each spin has a magnetic moment \( m = 2\mu S \) pointing in the same direction as the spin. The system is in an external magnetic field \( H \) in the z direction and is in equilibrium at a temperature \( T \). The Hamiltonian for the system is

\[
\mathcal{H} = J S_z(1) S_z(2) + J S_z(2) S_z(3) - 2\mu H [S_z(1) + S_z(2) + S_z(3)],
\]

where \( J \) and \( \mu \) are positive constants.

(a) List each of the possible microstates of the system and the corresponding energy. Sketch the energy-level diagram for the case where \( 2\mu H > J/2 > \mu H \) indicating any degeneracy.

(b) For each of the following conditions write down the limiting values for the internal energy \( U(T,H) \), the entropy \( S(T,H) \) and the magnetization \( M(T,H) \):

(i) \( T = 0 \) and \( H = 0 \)
(ii) \( T = 0 \) and \( 0 < H < J/\mu \)
(iii) \( T = 0 \) and \( J/\mu < H \)
(iv) \( J < kT \) and \( H = 0 \)

(c) On the basis of simple physical considerations, obtain the limiting temperature dependence at high and at low temperatures of the specific heat at constant field \( C_H(T,H) \) when \( H = 0 \). Sketch \( C_H \) vs. \( T \) for \( H = 0 \).

(d) Obtain the partition function for the system.

(e) Find the magnetization \( M(T,H) \) and show that, in the limit \( kT \gg \mu H \) or \( kT \gg J \), it reduces to \( M \approx 3\mu^2 H/kT \).
Consider a system of $N$ spin-$\frac{1}{2}$ magnetic moments $\vec{\mu}$ in thermal equilibrium at a temperature $T$ in a static magnetic field $\vec{B}$.

(a) Determine the canonical probability $p_+$ that any given $\vec{\mu}$ is aligned with $\vec{B}$, hence, in the ground state with energy $(-\varepsilon) = -\mu B$. Determine the alternate probability $p_-$ that $\vec{\mu}$ is anti-aligned with $\vec{B}$, hence, in the excited state with energy $\varepsilon = \mu B$, at given $T$.

(b) Show that the Energy-Temperature relation, or equation of state, is

$$E = -(N\varepsilon) \tanh(\beta\varepsilon),$$

where $\beta = \frac{1}{KT}$.

(c) Now suppose that $T < 0$, corresponding to a "population inversion." Why and how such an equilibrium state at a negative temperature may be achieved for this system?

Now that we have the $N$ spin-$\frac{1}{2}$ system at an initial negative temperature $T_i < 0$, we put it in thermal contact with a small "thermometer" at an initial temperature $T_i > 0$. Suppose that the "thermometer" is an "ordinary" system, made for instance with an ideal gas of $n$ molecules, obeying the equations of state $u = \frac{3}{2} nKT$ and $PV = nKT$. After thermal contact is established, the $N$ spin-$\frac{1}{2}$ system and the small thermometer reach a common final equilibrium temperature $T_f$.

(d) Determine the relation between the final $T_f$ and the initial $T_i < 0$, assuming $N >> n$. Consider in particular the limiting cases $|KT_i| << \varepsilon$ and $|KT_i| >> \varepsilon$. Why is the final $T_f$ always positive, with $KT_f >> \varepsilon$?

**Hint:** Use conservation of internal energy for the composite system before and after thermal contact is established between its two components.
Consider an ideal non-relativistic gas of \( N \) particles in a given volume \( V \), in the classical limit.

(a) Determine the canonical probability \( p(\varepsilon, T) \) \( d\varepsilon \) that any one particle has an energy between \( \varepsilon \) and \( \varepsilon + d\varepsilon \), assuming that the gas system is kept at a given temperature \( T \).

**Hint:** Remember that the degeneracy is \( g(\varepsilon) d\varepsilon = AV \varepsilon d\varepsilon \), and that \( p(\varepsilon, T) \) \( d\varepsilon \) must ultimately integrate to 1.

(b) The corresponding microcanonical probability is 

\[
p(\varepsilon, E) \, d\varepsilon = \frac{\Omega(N-1, E - \varepsilon)}{\Omega(N, E)} \cdot g(\varepsilon) d\varepsilon,
\]

assuming that the gas system is kept at a given energy \( E \), i.e. thermally isolated. Explain the physical meaning of all three factors comprised in the expression of \( p(\varepsilon, E) \) and its physical interpretation.

(c) Recall that \( \Omega(N, E) \propto E^{N-1} \) microcanonically. Then consider \( \log \{ p(\varepsilon, E) \} \) and expand it for \( N \gg 1 \) and \( E \gg \varepsilon \), without concern for additive constants independent of \( E \) and \( \varepsilon \).

Exponentiate the result to obtain \( p(\varepsilon, E) \) and fix a multiplicative constant by integrating \( p(\varepsilon, E) \, d\varepsilon \) to 1.

(d) For which relation between \( E \) and \( T \) does the microcanonical \( p(\varepsilon, E) \) coincide with the canonical \( p(\varepsilon, T) \)? Which Equations of State does that \( E-T \) relation represent in the microcanonical and canonical ensembles?

**Useful Formulas:**

\[
\Gamma(n) = (n-1)! = \int_0^\infty dx x^{n-1} e^{-x}, \quad \text{Re}(n) > 0.
\]

\[
\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.
\]

\[
\Gamma(n+1) = n\Gamma(n).
\]
For a ferromagnetic material in the absence of an applied field \((H=0)\), the spontaneous magnetization is a maximum at \(T=0\), decreases to zero at the critical temperature \(T = T_c\), and is zero for all \(T > T_c\). For temperatures just below \(T_C\) the magnetic susceptibility and the temperature coefficient of \(M\) might be modeled by the expressions

\[
X_T = \left( \frac{\partial M}{\partial H} \right)_T = \frac{a}{(1 - T/T_c)} + 3bH^2
\]

\[
\left( \frac{\partial M}{\partial T} \right)_H = \frac{1}{T_c} \left( \frac{f(H)}{\left(1 - \frac{T}{T_c}\right)^2} \right) - \frac{M_0}{2T_c} \left( \frac{1}{\left(1 - \frac{T}{T_c}\right)^{1/2}} \right)
\]

where \(M_0\), \(T_c\), \(a\) and \(b\) are constants and \(f(H)\) is a function of \(H\) alone with the property that \(f(H=0) = 0\).

(a) Find \(f(H)\) by using the fact that \(M\) is a state function.

(b) Find \(M(H,T)\).
Ph.D. Comprehensive Examination – Fall 2007
Statistical Mechanics 600-2

(a) Consider an ideal gas of \( N \) electrons in a volume \( V \). Calculate the Fermi energy \( \varepsilon_F \) in terms of \( N \) and \( V \).

Consider now a simple model of a metal that consists of a crystal lattice made up of \( N \) particles in a volume \( V \) and an ideal gas of \( N \) electrons confined to the same volume \( V \). In this model, all interactions between the electrons and between the electrons and the lattice are neglected. The specific heat \( C_V \) of the metal may be written as the sum of the lattice and electron gas contributions: \( C_V = C^L_V + C^e_V \).

(b) The following argument leads to a result that is inconsistent with experimental observations:

"By equipartition of energy, at room temperature \( C^L_V = 3Nk \) and \( C^e_V = 3Nk/2 \), therefore the metal has a specific heat that is \( \sim 50\% \) larger than that of an insulator (the crystal lattice with no free electron gas)."

Explain why this argument is incorrect.

(c) Make a rough estimate of the width of the Fermi function at a temperature \( T \), where \( T \) is of the order of room temperature. Use this result to make a rough, but much improved, estimate of \( C^e_V \). Explain how your result resolves the argument posed in part (b).

We now wish to make a more quantitative calculation of \( C^e_V \) for low temperatures \( T \ll T_F \), where \( T_F \) is the Fermi temperature), good to second order in \( T \). (See hint below.)

(d) Calculate the chemical potential \( \mu(T, V, N) \) of the free-electron gas to order \( T^2 \).

(e) Find the mean energy per particle \( \langle \varepsilon \rangle \) of the free-electron gas to order \( T^2 \).

Eliminate \( \mu \) from your result using the result from part (d). Show that you recover the familiar result \( \langle \varepsilon \rangle = \frac{3}{5} \varepsilon_F \) at \( T = 0 \).

(f) Find \( C^e_V \) in this same approximation. Discuss the relative contributions of \( C^L_V \) and \( C^e_V \) to \( C_V \) as \( T \rightarrow 0 \).

**Hint:** The following result may be useful for parts (d) and (e):

\[
\langle \varepsilon' \rangle = \frac{3}{2 \varepsilon_F^{3/2}} \int_0^\infty \frac{\varepsilon^{1/2} d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1} = \frac{3}{2} \frac{\mu^{3/2}}{\varepsilon_F^{3/2}} \left[ \pi^2 \frac{\mu^{1/2} (kT)^2}{4 \varepsilon_F^{3/2}} \right]
\]
Consider an ideal gas of $N$ non-relativistic fermions ($\Sigma=\frac{1}{2}$) in a three-dimensional volume $V$ at an absolute temperature $T$.

A. Derive or provide the expression for $n(\epsilon)d\epsilon$, representing the number of fermions with energy in the range $\epsilon$ to $\epsilon+d\epsilon$, in terms of the chemical potential $\mu=\mu(T,N/V)$.

B. Determine $n(\epsilon)$ for $T\to 0$.

C. Determine $\epsilon_f=\mu(T=0,N/V)$.

D. Sketch $n(\epsilon)$ for $kT<\epsilon_f$.

E. Determine $n(\epsilon)$ for $\exp(-\mu/kT)=\frac{1}{Z} \gg 1$ in terms of $N$ rather than $Z$.

F. Sketch $n(\epsilon)$ for $\exp(-\mu/kT)=\frac{1}{Z} \gg 1$.

G. Find the most probable energy $\epsilon^*$ for $T\to 0$. Find the corresponding maximum $n(\epsilon^*)$ in terms of $N$ and $\epsilon_f$.

H. Find the most probable energy $\epsilon^*$ for $\exp(-\mu/kT)=\frac{1}{Z} \gg 1$. Find the corresponding maximum $n(\epsilon^*)$ in terms of $N$ and $kT$.

I. Plot and compare on the same graph $n(\epsilon)$ vs. $\epsilon$ the two cases: $n(\epsilon)$ for $T=0$ and $n(\epsilon)$ for $T\gg \frac{\epsilon_f}{k}$. Which is larger and why?
Calculate the grand partition function for a system of $N$ non-interacting quantum mechanical harmonic oscillators with natural frequency $\omega_0$ for the following two cases:

(a) Bose statistics
(b) Boltzmann statistics.
(c) What is the probability that a given random oscillator is in the $n^{th}$ quantum state?
Consider a system of $N$ non-interacting spin-$3/2$ particles with magnetic moments $\mu_z = -g\mu_B J$, where $g = 2$, $\mu_B = \frac{\hbar}{2me}$, $J = -3/2, -1/2, 1/2, 3/2$. These particles are placed in a uniform magnetic field $\vec{B}$, which generates four energy levels, $\varepsilon = -3\varepsilon_B, -\varepsilon_B, \varepsilon_B, 3\varepsilon_B$, where $\varepsilon_B = \mu_B B$, for each particle. The system is maintained at a temperature $T$.

a) What are the canonical probabilities $p(\varepsilon)$ that any one particle is in each of the four $\varepsilon$-levels?

b) What is the average energy $\langle \varepsilon \rangle$ of any one particle?

c) Determine all four $p(\varepsilon)$ and $\langle \varepsilon \rangle$ to first order in $\beta = 1/kT$ for $kT >> \varepsilon_B$ and check the consistency of your results in this limit.

d) Determine the specific heat $C$ (per particle) for $kT >> \varepsilon_B$. Why does $C$ vanish with increasing $T$?

e) Determine all four $p(\varepsilon)$ and $\langle \varepsilon \rangle$ to first order in $e^{-\beta\varepsilon}$ for $kT << \varepsilon_B$.

f) Determine the specific heat $C$ for $kT << \varepsilon_B$. Why does $C$ vanish exponentially at low $T$?

g) Could a "population inversion" be realized for this system and why? If so, for which temperatures and how?
The Hamiltonian of a two-dimensional dipolar molecule of mass $m$, moment of inertia $I$, and dipole moment $\mu$, in an electric field $E$ is

$$H = \frac{1}{2m} (p_x^2 + p_y^2) + \frac{\mu^2}{2I} - \mu E \cos \theta$$

where $\theta$ is the angle between $\vec{E}$ and $\vec{\mu}$.

a) Find the canonical partition function of a system of $N$ such classical dipolar molecules confined to an area $A$.

b) Obtain low-temperature and high-temperature approximations for the specific heat at constant area, $C_A$, and confirm that the effect of the dipole-field interaction becomes negligible when $\mu E << kT$.

c) The probability of finding a two-dimensional dipolar molecule in a state described by the phase space variables $(x, y, p_x, p_y, \theta)$ is proportional to $e^{\beta H}$. Show that if the electric field is not uniform, then the gas is more dense where the field is higher.

Hints: The modified Bessel function $I_\nu(u)$ is a monotonically increasing function of $|u|$, with an integral representation:

$$I_\nu(u) = \frac{1}{\pi} \int_0^\pi e^{ucos \theta} d\theta$$

Its power series expansion is:

$$I_\nu(u) = \sum_{k=0}^\infty (u/2)^{2k} / (k!)^2,$$

while for large $u$:

$$I_\nu(u) \sim e^u / \sqrt{2\pi u}.$$
A cylinder of radius $R$ and length $L$ rotates about its axis with a constant angular velocity $\omega$. Ignore the effects of gravity and treat the system classically assuming that thermal equilibrium is established at temperature $T$.

(a) Obtain an expression for the density distribution of a classical ideal monatomic gas ($N$ atoms in equilibrium at temperature $T$) enclosed in the cylinder; that is, calculate the density $\rho$ as a function of $r$, where $r$ is the radial coordinate measured from the cylinder axis.

(b) What can you infer qualitatively from this about the distribution of gas in the cylinder?

**HINT:** The Hamiltonian that describes the system is $H = H_0 - \omega \mathcal{L}$ where $H_0$ is the Hamiltonian that one would obtain if the system were not rotating and

$$\mathcal{L} = \sum_{i=1}^{N} \left(x_i p_{yi} - y_i p_{xi}\right)$$

denotes the system’s angular momentum. In this expression $x_i$ and $p_{yi}$ are the $x$-component of the position coordinate and the $y$-component of momentum for the $i^{th}$ atom, respectively. Similarly, $y_i$ and $p_{xi}$ are the $y$-component of the position coordinate and the $x$-component of momentum for that atom. Apply the canonical distribution for $H$. 

Show that the internal energy of an ideal Bose gas in a volume $V$ at a temperature $T$ is given by

$$E = \frac{3}{2} V k T \left( \frac{2\pi m k T}{\hbar^2} \right)^{3/2} \sum_{n=1}^{\infty} \frac{e^{\mu / k T}}{n^{3/2}}$$

when the degeneracy is weak, that is, when the magnitude of the chemical potential $|\mu| \gg kT$ (recall that if the lowest value of the single particle energy is taken to be zero, then $\mu \leq 0$). The other symbols have their usual meanings: $m$ is the mass of one of the gas molecules, $k$ is Boltzmann's constant, and $\hbar$ is Planck's constant divided by $2\pi$.

To solve this problem, begin with the expression for the energy of an ideal Bose gas

$$E = \sum_s \frac{\epsilon_s}{e^{(\epsilon_s - \mu) / kT} - 1},$$

and convert the sum over states $s$ to an integral over energy as

$$\sum_s \rightarrow \int d\epsilon \; D(\epsilon),$$

where $D(\epsilon)$ is the density of states. Display $D(\epsilon)$ explicitly, and then use it to obtain $E$ as required.

**HINT 1:** The expansion $\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n$ for $|z| < 1$ may prove to be useful.

**HINT 2:**

$$\int_0^{\infty} dx \; x^{(m - \frac{1}{2})} e^{-x} = \Gamma \left( m + \frac{1}{2} \right) = \frac{1 \cdot 3 \cdot 5 \cdots (2m - 1)}{2^m} \sqrt{\pi}.$$
Black-body radiation may be viewed as an ideal gas of photons, with a number of photons

\[ N_e = \int_0^\infty \frac{g(\varepsilon)}{e^{\beta \varepsilon} - 1} \, d\varepsilon \]

in the excited states (or "gas phase"), and an undetermined number of photons in

the ground state with \( \varepsilon = 0 \) (the "condensed phase"). Equivalently, the two "phases" are in

equilibrium at \( z = e^{\mu/kT} = 1 \).

The density of states \( g(\varepsilon) \) reflects the energy-momentum relation \( \varepsilon = c|\vec{p}| \) for zero-mass

photons, as well as their \( g_\theta = 2 \) "spin degeneracy."

a) Determine \( g(\varepsilon) \) and \( N_e \), in terms of \( V, T \), and the constants \( \hbar, c \) and \( k \).

The pressure and energy of the photon gas can be obtained from the grand-canonical relations

\[ \frac{PV}{kT} = \log Z = \int_0^\infty g(\varepsilon) \log (1 - e^{-\beta \varepsilon}) \, d\varepsilon \]

and

\[ U = -\frac{\partial}{\partial \beta} (\log Z)_V = \int_0^\infty \frac{\varepsilon g(\varepsilon) \, d\varepsilon}{e^{\beta \varepsilon} - 1}. \]

b) Determine \( P \) and \( U \), and the relation between them, in terms of \( V, T \), and constants.

The equilibrium at \( z = 1 \) between the "gas phase" and the "condensed phase" is expected

to obey the Clapeyron equation, \( \frac{dP}{dT} = \frac{T \Delta s}{\Delta v} \).

c) Interpret and determine \( \Delta s \) and \( \Delta v \), and verify that the Clapeyron equation is obeyed by

the photon gas/condensate system.

Useful formulae:

\[ \int_0^\infty \frac{x^{n-1} \, dx}{e^x - 1} = \Gamma(n) \zeta(n), \quad \zeta(3) \approx 1.202, \quad \zeta(4) \approx 1.082. \]
Consider a system of $N$ hard-sphere particles of diameter $b$ that are constrained to move in one-dimension (i.e., along a line). Let the coordinate of the left side of the $i$-th particle be $x_i$. The length of the system is fixed such that $0 \leq x_i \leq L$ for all $i$.

(a) Obtain the partition function $Q_N(\beta, L)$ and from it, the Helmholtz free energy, the internal energy, and the heat capacity at constant length. [Note: Since the particles cannot pass through one another, they must be treated as if they are distinguishable.]

(b) Show that the equation of state for the pressure is the Tonks equation of state (Tonks, 1936):

$$p = \frac{NkT}{x_N - (N-1)b}$$

(c) Show that if a constant, infinite-ranged two-body attractive interaction $u_{ij} = -2a/(N-1)/L$ is added, the equation of state for the system becomes the one-dimensional van der Waals equation, for $N \gg 1$.

Now consider the original system (not the one in part (c)) but with the constraint of constant length replaced by a constraint of constant force $F$. Namely, the rightmost particle is subjected to a constant force $F$ towards the origin. The force is derivable from the potential $U = F(x_N - (N-1)b)$ such that $U=0$ corresponds to the lowest energy state of the system.

(d) Write the Hamiltonian for the system.

(e) Calculate the partition function $Y_M(\beta, F)$.

(f) Calculate the enthalpy and the heat capacity at constant force.

(g) Calculate the equilibrium position of the rightmost particle (the equilibrium length of the system). Show that this leads to the same equation of state as part (b). [Note that for a one-dimensional system, the pressure equals the force $F$; there is no "area" to divide by.]
Consider a system of $N$ particles placed along a one dimensional lattice, each particle being separated from its neighbor by a distance $a$. The particles can exist in one of two states: spin up (↑) or spin down (↓). One of the possible configurations of the lattice is shown below. Assume that $N \gg 1$.

By virtue of their spins each of the particles possesses a magnetic moment $\mu$ that is aligned with the direction of the spin.

**Part 1**
Suppose that the spins are non-interacting so that their directions are independent of one another (that is the direction of a given spin—up or down—has no influence on the orientation of any other spin).

(a) Write an expression for $P(n)$, the probability that exactly $n$ of the spins are in the “up” state, if $p$ is the probability that any given spin is in the “up” state. Note that $p$ is not necessarily $\frac{1}{2}$.

(b) What is $n_{\text{max}}$, the most probable value of $n$? What is $<n>$, the average value of $n$?

(c) What is the mean square fluctuation in $n$? That is, what is $<\Delta n^2> = <(n - <n>)^2>$?

**Part 2**
Suppose that an external magnetic field $B_0$ is now impressed on the system such that the spins are either aligned or anti-aligned with the field. If aligned the interaction energy is $-\mu B$, if anti-aligned $+\mu B$.

(d) Write an equation for the total energy of the system if $n$ of the $N$ spins are aligned with the field.

(e) Obtain the partition function of the system.

(f) Calculate the energy of the system as a function of temperature. Obtain expressions for the limiting forms of the energy at high and low temperatures. Sketch a graph showing the temperature dependence of the energy.

(g) Calculate the specific heat of the system as a function of temperature. Obtain expressions for the limiting forms of the specific heat at high and low temperatures. Sketch a graph showing the temperature dependence of the specific heat.

**Part 3**
Suppose that each spin interacts with its nearest neighbor such that there is a tendency for them to align with one another. That is suppose the interaction energy for adjacent spins is

$$\varepsilon = \begin{cases} 
- J & \text{(both spins up)} \\
- J & \text{(both spins down)} \\
+ J & \text{(one spin up, one spin down)} 
\end{cases}$$

This can be summarized by writing for a neighboring pair of spins, say, $i$ and $j$,

$$\varepsilon = -J \sigma_i \sigma_j$$

where $\sigma = +1$ if the spin is up and $\sigma = -1$ if the spin is down.

(h) Write an equation for the total energy of the $N$-particle system.

(i) Use this to write the partition function for the system as a sum over states.

(j) If the external magnetic field is zero, carry out the sum to obtain the partition function.
Consider a one-dimensional crystal of length $L$ made up of two types of atoms that alternate in the crystal. There are $N$ atoms of each type.

\[ \ldots \bullet \circ \bullet \circ \bullet \circ \bullet \circ \bullet \circ \bullet \circ \bullet \circ \bullet \circ \ldots \]

Assume that at low temperatures the system can be treated as a system of Bose-type elementary excitations or quasi-particles. Two types of quasi-particles must be considered; the dispersion relations for them are

Type 1: \[ \omega = ck \]

and

Type 2: \[ \omega = \omega_0. \]

In these relations $c$ and $\omega_0$ are constants. Compute the low temperature contributions to the specific heat from each type of excitation.

The following may be helpful:

\[ \int_0^\infty dx \frac{x}{e^x - 1} = \frac{\pi^2}{6} \quad \text{and} \quad \int_0^\infty dx \frac{x}{e^x + 1} = \frac{\pi^2}{12} \]
Consider a system of $N$ atoms in a magnetic field $H$ at temperature $T$. Each atom has a magnetic moment $\mu$ and each magnetic moment can point either parallel or antiparallel to the field. Assume that the atoms are fixed in a lattice and that the interaction between atoms is negligible.

a) Determine the average magnetic moment $\langle M \rangle$ in the system.

b) Determine the average of $(\delta M)^2$, where $\delta M = M - \langle M \rangle$ is the fluctuation of the total magnetic moment from its average value.

c) Derive a relation between your answer to part (b) and the susceptibility $\chi$, where

$$\chi = \left( \frac{\partial \langle M \rangle}{\partial H} \right)_{N,T}$$
Consider a dilute solution, i.e. a solution with low concentration \( c(n,N) = \frac{n}{N} \ll 1 \), in the gravitational field near the surface of the earth.

The free enthalpy of a dilute solution is given by:

\[
\Phi = N\mu_0 + n\alpha + nkT\ln(nl) = N\mu_0 + n(\alpha + kT\ln\frac{n}{e}) = N\mu_0(P,T) + nkT\ln\frac{c(n,N)}{e} + n\psi(P,T)
\]

Here \( \mu_0(P,T) \) is the chemical potential and \( N \) the number of molecules of the pure solvent, \( n \) the number of molecules of the solute, \( \alpha = f(P,T)/N \) describes the small change of enthalpy when a molecule of the solute is added to the solvent, and \( \psi(P,T) = kT\ln N e^{\frac{a}{kT}} \).

The chemical potentials of solvent (\( \mu \)) and solute (\( \mu' \)) in the external (gravitational) field are given by:

\[
\mu = \frac{\partial \Phi}{\partial N} + Mgz, \quad \mu' = \frac{\partial \Phi}{\partial n} + mgz
\]

where \( z \) is the height, \( M \) the molecular mass of the solvent, and \( m \) the molecular mass of the solute.

Obtain an expression for the change of concentration as a function of height by differentiating the equilibrium conditions \((T = \text{const}, \mu = \text{const}, \mu' = \text{const})\) with respect to the height.

Assume that \( \frac{dc}{dz} \) is small, i.e. it can be neglected, however not terms like \( \frac{1}{c} \frac{dc}{dz} \), and assume that the solution is incompressible, i.e. you may use the molecular volumes of solvent \( (V) \) and dissolved fluid \( (V') \) in your expression:

\[
V = \frac{\partial \mu_0}{\partial P} = \text{const}, \quad V' = \frac{\partial \psi}{\partial P} = \text{const}
\]
The object of this question is to calculate the translational, rotational and vibrational contributions to the specific heat of diatomic nitrogen gas for elevated temperatures, say temperatures above 300K. Specifically, consider a dilute gas of \( N \) molecules of \( N_2 \) confined to a volume \( V \) at temperature \( T \).

Begin by arguing that the translational and rotational degrees of freedom can be treated classically, but that the vibrational degree of freedom must be analyzed using quantum mechanics. Then calculate as a function of temperature

\[(a)\] the contribution to the specific heat arising from the translational degrees of freedom of the \( N_2 \) molecule;

\[(b)\] the contribution to the specific heat arising from the rotational degrees of freedom of the \( N_2 \) molecule; and

\[(c)\] the contribution to the specific heat arising from the vibrational degree of freedom of the \( N_2 \) molecule.

Data: For nitrogen

\[
\Theta_{\text{rot}} = \frac{\hbar^2}{2I k_B} = 2.847 \text{ K}
\]

where \( I \) is the moment of inertia of the nitrogen molecule about an axis through its center of mass and perpendicular to the molecule's symmetry axis, and \( k_B \) is Boltzmann's constant.

\[
\Theta_{\text{vib}} = \frac{\hbar \omega}{k_B} = 918.9 \text{ K}
\]

where \( \omega \) is the molecule's angular vibration frequency.
Consider a ultra-relativistic ideal Fermi gas in 2-dimensions. For each fermion, the energy-momentum relation then is \( \varepsilon = c \sqrt{p_x^2 + p_y^2} \). Suppose that there are \( N \) fermions confined to an area \( A \).

(a) \textbf{Determine the Fermi Momentum} \( p_{F_i} \) and from that the Fermi Energy \( \varepsilon_{F_i} \).
\textbf{Hint:} what is the area of the "Fermi ball" in 2-dimensions?

(b) \textbf{Determine the density of states} \( g(\varepsilon) \ d\varepsilon \) of the ultra-relativistic 2-dimensional ideal Fermi gas. Include a spin-degeneracy factor \( g_s = 2 \).

(c) \textbf{Verify that} \( N = \int_0^{\varepsilon_{F}} g(\varepsilon) d\varepsilon \) from (a) and (b).

(d) \textbf{Assume that the gas is kept at a temperature} \( T \ll \varepsilon_{F_i} / k_B \).
\textbf{Determine the Internal Energy} \( U = \int_0^{\varepsilon_{F}} \varepsilon \ g(\varepsilon) \ d\varepsilon \) in terms of \( N \) and \( \varepsilon_{F_i} \).

(e) \textbf{Show that} \( \frac{PA}{k_B T} = \frac{1}{2} U \) holds in fact at any temperature \( T \). Here the pressure \( P \) denotes the force per unit length exerted on the perimeter of the \( A \) area confining the gas.

\textbf{Hint:} recall that \( \frac{PA}{k_B T} = \log \mathcal{Z} = \int_0^\infty \log (1 + \varepsilon \ e^{-\beta \varepsilon}) \ g(\varepsilon) \ d\varepsilon \) and \( U = - \frac{\partial}{\partial \beta} (\log \mathcal{Z}) \ _{\varepsilon A} \). \textbf{Change variables of integration as} \( x = \beta \varepsilon \), in order to extract the relevant \( \beta \)-dependence.
Consider a system of $N$ quantum spin-$\frac{1}{2}$ magnetic moments in a magnetic field $\vec{B}$. Each magnetic moment has only two microstates with energies $-\varepsilon$ and $+\varepsilon$, where $\varepsilon = \mu_B B$. If a magnetic moment is aligned with $\vec{B}$, it has energy $-\varepsilon$ and there are $N_+$ such aligned moments. If a magnetic moment is anti-aligned with $\vec{B}$, it has energy $+\varepsilon$ and there are $N_-$ such anti-aligned moments. Now assume that $N = N_+ + N_-$ and $E = -\varepsilon N_+ + \varepsilon N_- = -\varepsilon m$, which is the total energy of the system, are all fixed, corresponding to a microcanonical ensemble description of the system of magnetic dipoles.

(a) Determine the number of microscopic states $\Omega(N, E) = \Omega(N_+, N_-)$ of the system of magnetic dipoles for a given $n$ and $E$, or, equivalently, for given $N_+$ and $N_-$, with $N_\pm = \frac{1}{2}(N \pm m)$.

(b) Determine the entropy $S(N, E)$ of the system of magnetic dipoles in the thermodynamic limit.

(c) Determine the equation of state that expresses $1/T$ as a function of $E$ and $N$. Invert it to solve for $E$ as a function of $N$ and $T$.

(d) Still in terms of given $N, N_+$, and $N_-$, what are the microcanonical probabilities $p_+$ and $p_-$ that any given magnetic dipole is aligned or anti-aligned with $\vec{B}$?

(e) Now determine the canonical probabilities $p_+(\beta)$ and $p_-(\beta)$ at given $\beta = 1/kT$ that any given magnetic dipole is either aligned or anti-aligned with $\vec{B}$.

(f) Equate the microcanonical and canonical probability ratio

$$\frac{p_+}{p_-} = \frac{p_+(\beta)}{p_-(\beta)}$$

to obtain the relation between $E$ and $T$, and compare it with that in part (c).
Consider an ideal gas of \( N \) indistinguishable particles of mass \( m \) confined to a volume \( V \) at a temperature \( T \). Using the classical approximation for the canonical partition function, \( z \), calculate the chemical potential \( \mu \) of the gas.

A gas of \( n \) indistinguishable classical particles, also of mass \( m \), is absorbed onto a surface of area \( A \) forming a two-dimensional ideal gas at temperature \( T \). The energy of an absorbed particle is \( \varepsilon = (p^2/2m) - \varepsilon_0 \), where the momentum \( p \) has components \( p_x \) and \( p_y \) and \( \varepsilon_0 \) is the surface binding energy. Calculate the classical canonical partition function \( z \) and the chemical potential \( \mu \) of this gas.

If at temperature \( T \) the particles on the surface and in the surrounding three-dimensional gas are in equilibrium, find the mean number of particles absorbed per unit area when the pressure of the three-dimensional gas is \( P \).

Now consider the two-dimensional gas as a system that can exchange particles with the three-dimensional gas. Obtain the grand canonical partition function for the two-dimensional gas by considering the number of configurations for distributing \( n \) identical particles on \( K \) surface sites. Hence, obtain the mean fraction of surface sites occupied \( \theta \) in terms of the pressure of the three-dimensional gas.
Consider a one-dimensional system of $N$ classical non-interacting particles subject to an external potential

$$V(z) = V_0 \log \left( 1 + \frac{|z|}{L} \right),$$

where $V_0$ and $L$ are constants.

a) Plot $V(z)$.

b) Calculate the single-particle partition function

$$Q_1(\beta) = \frac{1}{\hbar} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dp e^{-\beta H}$$

where $H(z, p_z) = \frac{p_z^2}{2m} + V(z)$ and $\beta = \frac{1}{kT}$. For which values of $kT$ is $0 \leq Q_1 < +\infty$?

[Hint: The change of variable $\zeta = 1 + z/L$ may be helpful in evaluating an integral.]

c) Calculate the average energy of the system $U = -NkT \frac{\partial}{\partial \beta} \left( \log Q_1 \right)$.

d) Referring back to your plot in part (a), What is the largest distance $z_c$ that a particle of energy $\epsilon$ can reach?

e) Is the particle confined by the potential $V(z)$ when $\epsilon \rightarrow +\infty$? Discuss why the system cannot be in thermodynamic equilibrium for $kT \geq V_0$. Compare this system with the situation of $V(z) = 0$ for all $-\infty < z < +\infty$, where there are no walls to confine the particles at all.
Consider a monoatomic isotropic two-dimensional square lattice of $N$ atoms with lattice spacing $a$. Assume Debye's model, with dispersion relation

$$ \omega = c \sqrt{k_x^2 + k_y^2}.$$ 

a) Determine the corresponding density of states $g(\omega) d\omega$. 

b) Determine the Debye cut-off frequency $\omega_D$. 

c) Determine the exact expression for the average energy $U$. 

d) Determine $U$ in the $kT \gg \hbar \omega_D$ limit, and compare your result with that of the classical equipartition theorem. 

e) Determine $U$ in the $kT \ll \hbar \omega_D$ limit.
Consider a system of \( N \) non-interacting particles in equilibrium at a temperature \( T \). The individual particles can have energy \( E = 0 \) or \( E = \varepsilon \).

(a) Obtain an expression for the internal energy of the system as a function of \( N \) and \( T \).

(b) Evaluate the specific heat. Make a graph showing this as a function of temperature. Work out the behavior of the specific heat in the limits \( T \to 0 \) and \( T \to \infty \).

(c) Suppose that instead of just 0 and \( \varepsilon \), the individual particles can have energies
\[ E = 0, \varepsilon, 2\varepsilon, \ldots n\varepsilon \ldots \]
How does this change the results obtained in part (b)?
A two-stranded DNA molecule can be modeled as a zipper with N links. Each link has a state in which it is closed with energy 0 and a state in which it is open with energy $\varepsilon$. We require, however, that the zipper can only unzip from the lower end, and that the link number $s$ can only open if all the links below it ($I, 2, ..., s-1$) are already open.

(a) Calculate the partition function $Z$ of the system at temperature $T$.

Hint: \[ \sum_{x'=0}^{N} x' = \frac{1-x^{N+1}}{1-x} \]

(b) In the limit $\varepsilon >> k_B T$, find the average number of open links.
Consider an ideal gas of HCl molecules in the classical limit for the translation degrees of freedom.

(a) Write the canonical partition function $Q_N(T,V)$ as a product of three factors originating from the translational, rotational, and vibrational degrees of freedom of the molecules.

(b) Determine the contribution of the translational degrees of freedom to the internal energy $U$, to the heat capacity $C_V$ at constant volume $V$, and to the gas pressure $P$.

(c) Determine the contribution of the rotational degrees of freedom to $U$ and $C_V$. Evaluate explicitly the single-particle partition function $j_{rot}(T)$ in the limits $T>>\Theta_r$ and $T<<\Theta_r$, where $\Theta_r = \frac{\hbar^2}{2Ik} \approx 15K$ ($I$ is the moment of inertia.) Determine the corresponding contributions to $U$ and $C_V$ in those limits.

(d) Determine the contribution of the vibrational degrees of freedom to $U$ and $C_V$. Evaluate $j_{vib}(T)$, $U_{vib}$, $(C_V)_{vib}$ exactly at all temperatures $T$, and then consider specifically the contributions in the $T>>\Theta_v$ and $T<<\Theta_v$ limits, where $\Theta_v = \frac{\hbar\omega}{k} \approx 10^3 K$.

(e) Sketch $\frac{C_V}{Nk}$ as a function of $T$ in the entire relevant temperature range.
Consider an ideal gas in the classical limit. The energy-momentum relation is \( \epsilon = B_s |\vec{p}|^s \), where, for example, \( s = 2 \) and \( s = 1 \) apply to the non-relativistic or ultra-relativistic limits respectively, and \( B_s \) is the corresponding appropriate constant \( \left( \frac{1}{2m} \right) \) or \( c \) for the same examples.

Suppose the ideal gas is spatially confined inside an \( n \)-dimensional "box" which is of length \( L \) in each dimension.

(a) Determine the single-particle density of states \( g(\epsilon) \, d\epsilon \) for arbitrary \( n \) and \( s \).

\[ \text{Hint: Recall that } g(\vec{p}) \, d^n \vec{p} = \frac{L^n}{h^n} \, d^n \vec{p}, \text{ and that the volume of a sphere of radius } p \text{ in } n \text{ dimensions is } V_n(p) = C_n p^n, \text{ where } C_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)}. \]

(b) Determine the canonical single-particle partition function \( Q_1(\beta, L) = \int_0^\infty e^{-\beta \epsilon} \, g(\epsilon) \, d\epsilon \) for arbitrary \( n \) and \( s \).

(c) Verify the previous general result for \( Q_1 \) in some of the familiar cases, where \( s = 2 \) or \( s = 1 \) and \( n = 3 \) or \( n = 2 \).
Consider a dilute gas of \( N \) molecules confined to a chamber of volume \( V \) where it is in equilibrium at some high temperature \( T \). Through a window in the chamber containing the gas, one observes a spectral line characteristic of some electronic transition. Because of the molecules' thermal motion the observed spectral line is Doppler broadened.

(a) Show that the intensity function describing the shape of the line as a function of frequency \( S(\omega) \) is a Gaussian function centered at \( \omega_0 \), the spectral frequency for a molecule at rest. Treat the translational degrees of freedom classically.

**HINT:** Recall that the Doppler shifted frequency \( \nu \) for a light source moving toward (-) or away from an observer (+) at a speed \( v \ll c \) is

\[
\omega = \frac{\omega_0}{1 \pm \frac{v}{c}},
\]

where \( \nu_0 \) is the frequency observed in the rest frame of the source.

(b) Obtain the temperature dependence of the width of the line.

Suppose that the gas is composed of rigid symmetric-top molecules (two equal moments of inertia). The rotational energy levels are given by

\[
e_{\lambda m} = \frac{\hbar^2}{2} \left[ \frac{I(j + 1)}{I_1} + \lambda^2 \left( \frac{1}{I_3} - \frac{1}{I_1} \right) \right] = \alpha j(j + 1) + \gamma \lambda^2
\]

where \( I_1, I_2 \) and \( I_3 \) are the principal moments of inertia and where the quantum numbers \( j, \lambda, \) and \( m \) take the values

\[
0 \leq j < \infty, \quad -j \leq \lambda < j, \quad -j \leq m \leq j.
\]

The energy is degenerate with respect to \( m \). Assume \( I_1 > I_3 \) so that \( \gamma > 0 \).

(c) Continuing to treat the translational motions classically, obtain the total partition function of the system in the form

\[
Z_N(V, T) = f(N, V, T) \left[ \sum_j \sum_\lambda g_{\lambda}(\alpha, \beta, \gamma) \right]^N
\]

where \( \beta = 1/kT \) and \( f \) and \( g \) are functions that you must determine. Here \( f(N, V, T) \) is the translational partition function and the other factor is the rotational partition function.

(d) From this obtain the specific heat of the gas in the limit of high temperatures.

**HINT:** For part (d) use the Euler-Maclaurin formula

\[
\sum_{m=1}^{n} F_1 = \int_{1}^{n} dx F(x) + \left[ F(n) + F(m) \right] \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} B_k \left[ F^{(2k-1)}(n) - F^{(2k-1)}(m) \right]
\]

where \( F^{(\mu)} \) denotes the \( \mu \)th derivative of \( F \) and the \( B_k \) are the Bernoulli numbers. Retain only the first term on the right and try integrating by parts.
The Debye theory treats a crystal as a continuum and hence the phonon dispersion relation is 
\[ \omega = ck, \] where \( c \) is the speed of sound in the body. In a ferromagnetic solid at low temperatures, 
there also exist quantized waves of magnetization (magnons) for which the dispersion relation is of 
the form \[ \omega = bk^2, \] where \( b \) is a constant. In both these cases a large-\( k \) (small wavelength) cutoff 
must be imposed; take the cutoffs in the two cases as \( k_D \) and \( k_M \), respectively.

Obtain expressions valid near \( T = 0 \) for the phonon and magnon contributions to the specific heat. 
Your answers may include a dimensionless multiplicative constant in the form of an integral that you 
need not evaluate. The main point of this problem is to obtain the temperature dependence of the two 
contributions to the specific heat at low temperatures.
In this problem we study the properties of the surface of the model two-dimensional "liquid" shown in the figure below. The configurations of the surface are confined to a triangular lattice and the ends of the surface are pinned at the two points shown, which are a distance $L$ apart. Assume that "overhangs" are forbidden so that, starting from one pin, every step along the surface is taken towards the other pin. Now consider the length of the surface. Take the zero-energy configuration to be the flat surface (of length $L$) and the energy required to increase the length by one lattice spacing $a$ to be $\epsilon$.

a) Write out the combined first and second laws of thermodynamics as applied to this system, ignoring $pV$ work.

b) Use the microcanonical ensemble to calculate the length of the surface as a function of temperature, $l(T)$.

c) Similarly, calculate the (one-dimensional) surface tension as a function of temperature $\sigma(T)$ and show that it is properly an intensive variable. Is this a good model for the surface tension of a liquid?
Consider a gas of Cl₂ chloride, it consists of diatomic molecules.

(1) Calculate the gas specific heat, \( C_p \), for a temperature range where the rotation, and translation degrees of freedom behave classically and the vibrational behave quantum mechanically.

(2) Derive the Debye specific heat for a low temperature monoatomic solid.

(3) Modify the Debye equation for the diatomic molecular chlorine solid.